A Lyapunov Function Approach to Energy Based Model Reduction
Menu

• Why model reduction?
• Lyapunov functions
• Mass spring damper example
• Tie into Sam’s talk
Model Reduction

- Automated model reduction techniques create complex models
- Engineers are lazy
- Heuristic model reduction requires experience, and
- It is difficult or impossible to quantify the reduction tradeoffs
Lyapunov Functions

• Are our friends
• Lyapunov (Complicated System ) = Scaler Value

\[ f(\bullet): \mathbb{R}^k \rightarrow \mathbb{R} \]

\[ E = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x^2 \]
Lyapunov Functions Are Bowls
Vector Norms
Canonical Example
Pendulum Example
Problem: Negative Semidefinite
Some Math

$$\frac{1}{2} m \ddot{x}^2 + \frac{1}{2} k x^2 = [x \quad \dot{x}] \begin{bmatrix} k & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$$

$$\dot{E} = kx\ddot{x} + m\dddot{x} = -bx = [x \quad \dot{x}] \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$$
Converse Lyapunov: Look for an Exponential

\[ W = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x^2 + \varepsilon x \dot{x} \]

\[ \varepsilon = \frac{b}{2} \hspace{1cm} \alpha = \frac{b}{m} \]

\[ \dot{W} = -\alpha W \]
“Mostly” Energy Function
Nonlinear Systems

• Nonlinear systems are more difficult to analyze
• Is there a physical generalization from linear to nonlinear?
Sam’s Talk

• Quantifies the reduction for damped mechanical systems

=> Bounded Force, Bounded State

• This generates a constant ellipsoid of possible state deviation from the simplified model
Bounded Input, Bounded State
Recap & Conclusion

• Automated models are too complex
• Lyapunov functionals are our friends
• Energy is not an ideal Lyapunov function, but it can be “tweaked” in the linear case
• Math is actually cool