CONTRACTION AND SUM OF SQUARES ANALYSIS OF HCCI ENGINES

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Abstract: By modulating engine valves to reinduct hot exhaust gas together with air and fuel, a clean and efficient form of autoignition can be created. Control of this combustion process, known as homogeneous charge compression ignition (HCCI), requires not only precise valve control but also a combustion control strategy that accounts for the cycle-to-cycle coupling through the exhaust. This paper outlines approaches for proving closed-loop stability of a valve controller and combustion controller using nonlinear analysis tools. Stability of the valve controller is shown using contraction analysis. Stability of the combustion controller is shown using sum of squares decomposition, convex optimization and the Positivstellensatz.

Keywords: internal combustion engines, engine control, model-based control, nonlinear analysis, nonlinear control systems, control system analysis, convex optimization, variable valve timing control

1. INTRODUCTION

Residual-affected homogeneous charge compression ignition holds great promise as a means to reduce NO_x emissions and increase efficiency in internal combustion engines (Caton *et al.*, 2003). During an engine cycle, fresh reactant charge and exhaust products from the previous cycle are inducted through the intake and exhaust valves, respectively. This gas mixture is then compression ignited, expanded and finally exhausted. The process therefore exhibits some fundamental control challenges. Unlike spark ignited or Diesel engines, where the combustion is initiated via spark and fuel injection, respectively, HCCI has no specific event that initiates combustion. Since exhaust gas is re-inducted from the previous cycle, cyclic coupling exists.

To influence the flows through the valves, precise control of a variable valve actuation (VVA) system is required. This paper briefly outlines an approach for electro-hydraulic valve system (EHVS) control that incorporates a full nonlinear model of the relevant physics in the control design. Since only valve position is considered measurable, a nonlinear controller/observer pair is designed for EHVS control. The stability of the approach follows from contraction analysis (Lohmiller and Slotine, 1998; Lohmiller and Slotine, 2000).

With precise, stable control of the valves, the next stability question is that of the combustion process itself. To address the issues of cyclic coupling and lack of a combustion trigger, a low-order nonlinear model of the HCCI combustion process was developed in previous work (Shaver and Gerdes, 2003). The model links together discretized versions of the various processes which occur during an HCCI engine cycle. What results is a physics-based model of HCCI combus-

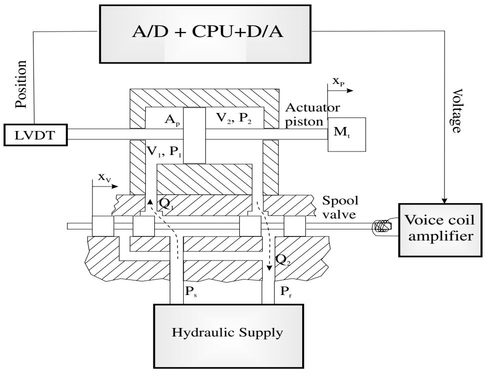

Fig. 1. EHVS schematic

tion, with peak in-cylinder pressure as an output, and inducted gas composition as an input. The model can be linearized about an operating point and used for controller synthesis. In this work, a Lyapunov-based analysis utilizing sum of squares decomposition and a powerful theorem from real algebraic geometry, the Positivstellensatz, estimates the domain of attraction for the non-linear system with an LQR controller. The resulting region of attraction proves stability of the system over most of the desired portion of the state space. Physically, this means that the LQR control stabilizes the nonlinear system over the majority of the desired operating range of HCCI.

2. EHVS SYSTEM MODEL

In this section, the complete model for the EHVS system is presented. The model employs well-known physical quantities and relationships, and for further details on the derivation the reader is directed to a standard reference (Merritt, 1967). Overall, the EHVS system model consists of three major blocks representing the dynamics of the spool valve position, the hydraulic pressures on the piston actuator, and the piston actuator position, respectively.

Applying the conservation of mass to each side of the actuator chambers we obtain

$$Q_1 = \frac{dV_{p_1}}{dt} + \frac{V_{p_1}}{\beta_e}\frac{dP_1}{dt} + C_{i_p}(P_1 - P_2) + C_{e_p}P_1 \quad (1)$$

$$-Q_2 = \frac{dV_{p_2}}{dt} + \frac{V_{p_2}}{\beta_e}\frac{dP_2}{dt} - C_{i_p}(P_1 - P_2) + C_{e_p}P_2 \quad (2)$$

where $V_{p_1} = V_{o1} + A_p x_p$, $V_{p_2} = V_{o2} - A_p x_p$,;

Q_1, Q_2 are the flows through the orifice to and from left and right chambers, respectively, as indicated in Fig 1; β_e is the lumped bulk compressibility modulus of the hydraulic oil and surrounding mechanical structure; C_{i_p} and C_{e_p} are the actuator internal and external leakage coefficients; V_{p_1} and V_{p_2} are the volumes of the left and right chambers, respectively; V_{01} and V_{02} are the volumes when $x = 0$, and A_p is the piston cross-sectional area. The momentum equation for the actuator piston and engine valve yields

$$M_t\frac{d^2x_p}{dt^2} = A_p(P_1 - P_2) - (F_o + Kx_p) - B\frac{dx_p}{dt}, \quad (3)$$

where M_t is the combined actuator and engine valve mass, F_o is the spring pre-load force, K is the spring stiffness coefficient, B is the viscous damping, and x_p is the actuator displacement.

The spool valve response to an electrical input current can be described as a second order system

$$\frac{d^2x_v}{dt^2} + 2\zeta\omega_n\frac{dx_v}{dt} + {\omega_n}^2 x_v = \frac{K_{sc}}{K_f}{\omega_n}^2 i, \quad (4)$$

where ω_n is the spool valve natural frequency, ζ is the spool valve damping coefficient, K_{sc} is the spool valve coil gain, K_f the spool valve spring constant, and i is the input current.

Combining eqs. (1)-(4) we obtain the following nonlinear six state model of the valve dynamics

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = a_1 x_z + f_2(x_1, x_2) \quad (5)$$

$$\dot{x}_3 = V_1(x_1)\left(Q_1(x_3, x_5) - f_3(x_2, x_3, x_4)\right) \quad (6)$$

$$\dot{x}_4 = V_2(x_1)\left(Q_2(x_4, x_5) - f_4(x_2, x_3, x_4)\right) \quad (7)$$

$$\dot{x}_5 = x_6, \quad \dot{x}_6 = a_2 u + f_6(x_5, x_6) \quad (8)$$

where x_1 and x_2 are the piston position and velocity, x_3 and x_4 are the pressures P_1 and P_2, x_5 and x_6 are the spool valve position and velocity and u is the input current, and

$$x_z = x_3 - x_4, \quad a_1 = \frac{A_p}{M_t}, \quad a_2 = \frac{K_{ss}}{K_f}\omega_n^2,$$

$$V_1(x_1) = \frac{\beta_e}{V_{o1} + A_p x_1}, \; V_2(x_1) = \frac{-\beta_e}{V_{o2} - A_p x_1} \quad (9)$$

$$f_2(x_1, x_2) = -\frac{1}{M_t}(F_0 + kx_1) - \frac{B}{M_t}x_2 \quad (10)$$

$$f_3(x_2, x_3, x_4) = A_p x_2 + C_{ip} x_z + C_{ep} x_3 \quad (11)$$

$$f_4(x_2, x_3, x_4) = A_p x_2 + C_{ip} x_z - C_{ep} x_4 \quad (12)$$

$$f_6(x_5, x_6) = -\omega_n^2 x_5 - 2\zeta\omega_n x_6 \quad (13)$$

The tracking control design for the system given by (5)-(8) is discussed in the next section.

3. VVA SYSTEM CONTROL DESIGN

In the previous section, we have derived a physical model of the VVA behavior. In this section, we present the control structure for the VVA that requires only measurement of the valve position and uses information about the desired valve trajectory as input. The control structure consists of a nonlinear observer and controller. These two components of the control structure exploit the contractability properties (see (Lohmiller and Slotine, 1998),(Lohmiller and Slotine, 2000)) of the system and utilize a triangular structures approach (Seto *et al.*, 1994) to synthesize the appropriate control action. For brevity, we only present the control structure here and a qualitative outline of the proof of stability. The proposed control design has been verified in numerical simulations for tracking sinusoidal output profiles of practical interest up to a frequency of 200Hz. Further details can be found in (Kojić *et al.*, 2003).

The observer is given as follows:

$$\dot{\hat{x}}_1 = \hat{x}_2 - c_1(\hat{x}_1 - x_1) \quad (14)$$

$$\dot{\hat{x}}_2 = a_1\hat{x}_z + f_2(x_1, \hat{x}_2) - c_2(\hat{x}_1 - x_1) \quad (15)$$

$$\dot{\hat{x}}_3 = V_1(x_1)(Q_1(\hat{x}_3, \hat{x}_5) - f_3(\hat{x}_2, \hat{x}_3, \hat{x}_4)) \quad (16)$$

$$\dot{\hat{x}}_4 = V_2(x_1)\left(Q_2(\hat{x}_4, \hat{x}_5) - f_4(\hat{x}_2, \hat{x}_3, \hat{x}_4)\right) \quad (17)$$

$$\dot{\hat{x}}_5 = \hat{x}_6 \quad (18)$$

$$\dot{\hat{x}}_6 = a_2 u + f_6(\hat{x}_5, \hat{x}_6) \quad (19)$$

where the quantities denoted with $\hat{\mathbf{x}} = [\hat{x}_1, \ldots, \hat{x}_6]$ represent the estimates of the state vector, c_1,c_2 are positive constants, and u is the control input to be determined later.

Theorem 1: Let $\widetilde{\mathbf{x}} = \hat{\mathbf{x}} - \mathbf{x}$, and let $\dot{\hat{\mathbf{x}}}, \dot{\mathbf{x}}$ be given by (14)-(19), (5)-(8), respectively. Then, $\widetilde{\mathbf{x}}$ converges to zero exponentially.

Proof: Qualitatively, the proof demonstrates that the system described by (14)-(19) is contracting according to (Lohmiller and Slotine, 1998). Contraction implies that the system is "memory-less" with respect to its initial conditions when driven by a specified input. In particular, this means that after a certain transient period during which the states remain bounded, the behavior of a contracting system is solely determined by the input. Intuitively, this is indeed the case for practical EHVS systems under consideration here.

The observer in (14)-(19) consists of a cascade connection of a linear, (18)-(19), and a nonlinear system, (14)-(17). The linear subsystem is exponentially stable, and hence contracting by definition. It can also be shown that the nonlinearities introduced by the functions Q_1, Q_2 satisfy the contraction property $\frac{\partial Q_j}{\partial x_{j+2}} < 0, j = 1, 2$. Furthermore, for valving profiles of practical interest, which consist of sinusoids with a frequency of up to 200Hz, it can be shown that the overall nonlinear subsystem is contracting. Thus, the observer consisting of a cascade connection of two contracting systems is also contracting. □

We now proceed to design a the control law $u(\hat{\mathbf{x}}, t)$ such that x_1 tracks an arbitrary trajectory y_d. We assume that the trajectory y_d is smooth, and that higher derivatives $y_d^{(i)}$, $i = 1, \ldots, 5$ are known.

The proposed controller structure is given below

$$\hat{x}_z = \hat{x}_3 - \hat{x}_4, e_1 = \hat{x}_1 - y_d, e_z = a_1(\hat{x}_3 + \alpha) \quad (20)$$

$$e_5 = a_1(V_1 Q_1(\hat{x_{3,5}}) - V_2 Q_2(\hat{x_{4,5}})) + \dot{e_1} + \beta, \quad (21)$$

$$u = -\frac{f_6}{a_2} - \Gamma_6^{-1}\left(\gamma + \dot{e_z} + c_5 e_5 + c_6 \dot{e_5}\right) \quad (22)$$

where c_5, c_6 are positive constants, and the functions α, β, γ are chosen such that

$$\dot{e_1}\,(\alpha - f_2(x_1, \hat{x}_2) + \ddot{y_d}) \geq 0 \quad (23)$$

$$e_z\big(\beta + V_1 f_3(\hat{x_{2,3,4}}) - V_2 f_4(\hat{x_{2,3,4}}) + \dot{\alpha}\big) \geq 0 \quad (24)$$

$$\dot{e_5}\left(\gamma - \dot{\Gamma}_6\,\hat{x}_6 - \xi^T J_r \xi - r^T\,\dot{\xi} + e_1^{(3)} - \ddot{\beta}\right) \geq 0 \quad (25)$$

with

$$\Gamma_6 = a_1\left[V_1 \frac{\partial Q_1}{\partial \hat{x}_5} + V_2 \frac{\partial Q_2}{\partial \hat{x}_5}\right] \quad (26)$$

$$r_i = \frac{\partial}{\hat{x}_i}(V_1 Q_1 + V_2 Q_2),\ \ i=1, \ldots, 4;\ \ r_5=0 \quad (27)$$

$$J_{r_{i,j}} = \frac{\partial r_i}{\partial \hat{x_j}}\ \ i, j = 1, \ldots, 5 \quad (28)$$

$$\xi_i = \dot{\hat{x}}_i,\ \ i = 2, \ldots, 5 \quad (29)$$

It should be noted that the time derivatives of the defined quantities in (20)-(29) (such as e_1,e_z, etc) are taken along the trajectories of the observer system defined by (14)-(19). Hence, these time derivatives (such as $\dot{e_1}$,$\dot{e_z}$,etc) can be calculated on-line by the controller.

Theorem 2: For the system in (5)-(8), the observer and controller given by (14)-(19), (20)-(29), assure that x_1 tracks $y_d(t)$, and that all the states are bounded.

Proof: From (20)-(29), (14)-(19) we obtain

$$\ddot{e_1} = e_z - \alpha + f_1 - \ddot{y_d}, \quad (30)$$

$$\dot{e_z} = e_5 - \dot{e_1} - \beta + \dot{\alpha} - a_1(V_1 f_3 - V_2 f_4), \quad (31)$$

$$\ddot{e_5} = -\dot{e_z} - c_5 e_5 - c_6 \dot{e_5}, \quad (32)$$

From (30)-(32) it follows that the quantities e_1,e_z,e_5 are bounded and asymptotically converge to zero. From Theorem 1 we have that $\mathbf{x}$ converges to $\hat{\mathbf{x}}$, and hence we have tracking, and also that $\mathbf{x}$ is bounded. □

4. HCCI ENGINE MODEL

Having shown that system stability is assured for the VVA controller, the focus now turns to the stability analysis of the combustion process itself. By linking the distinct processes which occur during HCCI combustion, a nonlinear relation between the input to the system at engine cycle k (the molar ratio of re-inducted product to inducted reactant, α_k) and the measurable output at engine cycle k (peak in-cylinder pressure, P_k) was developed in previous work (Shaver and Gerdes, 2003):

$$P_k = \frac{d_1^k}{1+\alpha_k} \frac{c_{12}d_1^{k-1}+c_{13}d_2^{k-1}+c_{15}\chi\alpha_k P_{k-1}^{1/\gamma}}{c_{11}d_1^{k-1}+c_{10}d_2^{k-1}+c_{14}\chi\alpha_k P_{k-1}^{1/\gamma}} \quad (33)$$

where:

$$d\,_1^i = c_1 + c_2\alpha_i, \;\; d_2^i = P_{k-1}(1+\alpha_i) \quad (34)$$

The constants in this expression are directly related to physical quantities such as reactant and product specific heats, cylinder volumes where certain events take place (i.e. combustion, intake and exhaust valve closings and opening), inlet reactant temperature and intake pressure. For details see (Shaver and Gerdes, 2003). Note the presence of cycle-to-cycle dynamics, as the peak pressure on the current cycle, P_k, depends not only on the current cycles input, α_k, but also on the previous cycles input, α_{k-1}, and peak pressure, P_{k-1}.

From the non-linear model of HCCI combustion, a variety of different controllers can be synthesized to track the desired in-cylinder peak pressure. For instance, an LQR control law can be synthesized from a linearization of the system about an operating point $(\bar{\alpha}, \bar{P})$. The LQR controller takes the form:

$$\tilde{\alpha_k} = K_1\beta_{k-1} + K_2\tilde{\alpha}_{k-1} \quad (35)$$

where $\beta_k = (P_k - \bar{P})/\bar{P}$, and $\tilde{\alpha_k} = \alpha_k - \bar{\alpha}$. Figure 2 shows the structure of the closed-loop controller, which has been successfully implemented on an experimental test bed. For details on the α to IVO/EVC map, see (Shaver and Gerdes, 2003).

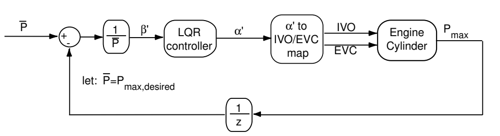

Fig. 2. Block diagram of controller implementation

5. ESTIMATING THE DOMAIN OF ATTRACTION USING THE SUM OF SQUARES DECOMPOSITION

The next step is showing the stability of the closed-loop system. The nonlinear dynamics as stated in Equations 33 and 35 are amenable to stability analysis using sum of squares (SOS) techniques. In particular, stability in the sense of Lyapunov will be shown for a candidate Lyapunov function. Additionally, the domain of attraction is estimated.

5.1 Problem Formulation

In this section, a methodology, outlined in (Jarvis-Wloszek, 2003; Papachristodoulou and Prajna, 2002), for finding the domain of attraction for systems of a specific form is presented. Consider the discrete nonlinear system:

$$x_k = f(x_{k-1}, u_{k-1}) \quad (36)$$

with the following equality constraints:

$$e_i(x_{k-1}, u_{k-1}) = 0, \quad for \;\; i = 1, ..., N_e \quad (37)$$

where $x \in \Re^n$ is the state of the system, and $u \in \Re^m$ is a collection of auxiliary variables (such as non-polynomial functions of states, uncertain parameters, etc.). We assume that the e_i's are polynomial functions in (x, u) and $f(x, u)$ is a vector of polynomial or rational functions in (x, u) with $f(0) = 0$.

If we can find a function of the states $V(x_{k-1})$ with $V(0) = 0$, $\Psi > 0$ and $\Delta V(x) \equiv V_k(x) - V_{k-1}(x)$, such that:

$$\{x \in \Re^n | V \le \Psi\} \backslash \{0\} \subseteq \{x \in R^n | \Delta V < 0\} \quad (38)$$

$$\{x \in \Re^n | V \le \Psi\} \backslash \{0\} \subseteq \{x \in R^n | V > 0\} \quad (39)$$

then the system is asymptotically stable about the fixed point $x = 0$ and has a region of attraction which includes $D = \{x \in \Re^n | V(x) \le \Psi\}$. To see this, just note that for the above to hold, $V_k(x) - V_{k-1}(x)$ must be negative in the set D. This proves that D is a region of attraction since a positive function of the states continually decreases within the set D.

We can pose the problem of showing feasibility of the candidate Lyapunov function and finding the largest value of Ψ (thus maximizing the stability region defined by D) as an optimization problem of the form:

$$\max_{V \in R_n, V(0)=0} \quad \Psi$$

$$s.t.$$

$$\{x \in \Re^n | V \le \Psi, x \ne 0, w\Delta V \ge 0\} = \phi \quad (40)$$

$$\{x \in \Re^n | V \le \Psi, x \ne 0, V \le 0\} = \phi \quad (41)$$

where $w(x, u)$ is a positive polynomial. This is just a set emptiness form of the constraints given in Equations 38 and 39. The inclusion of $w(x, u)$ will become clear at the end of the next section.

5.2 The Positivstellensatz

By utilizing a powerful theorem from real algebraic geometry, called the Positivstellensatz, the above optimization problem can be re-cast.

Theorem 3: The Positivstellensatz *Given sets of polynomials $f_1, ..., f_r$, $g_1, ..., g_t$, and $h_1, ..., h_u$ in R_{n+m}, the following are equivalent:*

1.) The set, $(x, u) \in \Re^{n+m}$ for which:

$$f_1(x,u) \geq 0, ..., f_r(x,u) \geq 0 \quad (42)$$
$$g_1(x,u) \neq 0, ..., g_t(x,u) \neq 0 \quad (43)$$
$$h_1(x,u) = 0, ..., h_u(x,u) = 0 \quad (44)$$

is empty.

2.) There exist $k_1, ..., k_u \in Z_+$, $s_i \in \Sigma_n$, $r_k \in R_{n+m}$ *such that*

$$f + g^2 + h = 0 \quad (45)$$

where: (46)

$$f = s_0 + \sum_{i_1=1}^{r} s_{i_1} f_{i_1} + \cdots + \sum_{i_1=1}^{r} \cdots \sum_{i_r=1}^{r} s_{i_1 \cdots i_r} f_{i_1} \cdots f_{i_r} \quad (47)$$

$$g = \left(g_1^{k_1} \cdots g_u^{k_u} \right), \; h = \sum_{k=1}^{u} h_k r_k \quad (48)$$

Proof: See (Bocknak *et al.*, 1986) Theorem 4.2.2 □

The set of multivariate polynomials in $(x,u) \in \Re^{n+m}$ is denoted above as R_{n+m}, while Σ_{n+m} represents the set of sum of squares polynomials. A multivariate polynomial $s(x,u)$ is sum of squares (SOS) if there exist polynomials $p_1(x,u), ..., p_q(x,u) \in R_{n+m}$ such that $s(x,u) = \sum_{i=1}^{q} p_i^2(x,u)$.

Note that there is no upper bound given on the required polynomial degrees of the s's and r's and the value of the k's that might be needed to satisfy Equation 45 if Equations 42-44 hold. However, there are finite degrees that will.

By recognizing the correspondence between Equations 37,40-41 and Equations 42 and 44, the domain of attraction problem can be re-cast as:

$$\max_{s_i \in \Sigma_n, V \in R_n, V(0)=0} \Psi$$
$$s.t.$$
$$s_1 + (\Psi - V)s_2 - V s_3 - V(\Psi - V)s_4 + \sum_{k=1}^{N_e} r_k e_k + L_1^2 = 0 \quad (49)$$
$$s_5 + (\Psi - V)s_6 + w\Delta V s_7 + w\Delta V(\Psi - V)s_8 + \sum_{k=1}^{N_e} r_k e_k + L_2^2 = 0 \quad (50)$$

where the $L_i(x)$s take the form $(x_1^{k_{1,i}} \cdots x_n^{k_{n,i}})$. For the case where $f(x_{k-1}, u_{k-1})$ is a rational vector field, such as $\frac{n(x,u)}{d(x,u)}$, the multiplier $w(x,u) > 0$ should be chosen so that $w(x,u)\Delta V(x)$ is a polynomial. Obviously $w(x,u)$ can be chosen as the denominator of $\Delta V(x)$ as long as it is always positive in the region of interest. For more detail see (Papachristodoulou and Prajna, 2002; Jarvis-Wloszek, 2003).

5.3 Sum of Square Programs

The above optimization can be cast as a sum of squares Program (SOSP). Sum of squares programs are formulated as follows:

Find the coefficients of:

$$\text{polynomials} \quad r_i(x), \text{for } i = 1, 2, ..., N_1 \quad (51)$$
$$\text{sum of squares} \quad s_j(x), \text{for } j = 1, 2, ..., N_2 \quad (52)$$

with a pre-defined structure, such that:

$$a_k(x) + \sum_{i=1}^{N_1} r_i(x) b_{i,k}(x) + \sum_{i=1}^{N_2} s_j(x) c_{j,k}(x) = 0 \quad (53)$$

for $k = 1, 2, ..., N_3$, where: $a_k(x)$, $b_{i,k}$ and $c_{j,k}$ are constant coefficient polynomials. Using this formulation, the domain of attraction problem as given at the end of Section 5.2, for a given value of Ψ, can be written as a sum of squares program: find the coefficients of sum of squares polynomials, $s_j(x,u), \text{for} j = 1, 2, ..., 8$, and polynomials, $r_i(x,u), \text{for} j = 1, 2, ..., N_e$, with a pre-defined structure, such that Equations 49 and 50 hold.

The candidate polynomial Lyapunov function $V(x)$ is chosen as part of the design process. As long as the chosen degrees of the s and r polynomials are large enough, the SOS program will show feasibility of Equations 49 and 50 if Equations 38 and 39 hold, for a given value of Ψ. A linesearch of Ψ can be made to find the largest Ψ, Ψ_{max}, that satisfies Equations 49 and 50. This will yield the estimate of the domain of attraction as $\{x \in \Re^n | V(x) < \Psi_{max}\}$, for a given $V(x)$. Other polynomial Lyapunov functions can be chosen to search for larger domains of attraction.

Sum of squares programs can be solved using convex optimization, in particular semi-definite programming (SDP) (Parrilo and Lall, 2003). Feasibility of a candidate Lyapunov function and an estimate for the domain of attraction can therefore be obtained efficiently from SDP.

6. DOMAIN OF ATTRACTION FOR THE HCCI SYSTEM

Re-writing the states of the system as: $x_{1,k-1} = \beta_{k-1}, x_{2,k-1} = \tilde{\alpha}_{k-1}$ and an auxiliary variable $u_{1,k-1} = (P_{k-1}^{\frac{1}{\gamma}} - \bar{P}^{\frac{1}{\gamma}})/\bar{P}^{\frac{1}{\gamma}}$. Equations 33- 35 can be re-written in form given in Equations 36-37, as:

$$x_{1,k} = \frac{c_1 + c_2(\bar{\alpha} + x_{2,k})}{\bar{P}(1 + (\bar{\alpha} + x_{2,k}))} \text{x}$$
$$\left(\frac{c_{12} d_1^{k-1} + c_{13} d_2^{k-1} + c_{15} \chi(\bar{\alpha} + x_{2,k}) \bar{P}^{\frac{1}{\gamma}} (1 + u_{1,k-1})}{c_{11} d_1^{k-1} + c_{10} d_2^{k-1} + c_{14} \chi(\bar{\alpha} + x_{2,k}) \bar{P}^{\frac{1}{\gamma}} (1 + u_{1,k-1})} - 1 \right)$$
$$x_{2,k} = K_1 x_{1,k-1} + K_2 x_{2,k-1}$$
$$e_1(x_{k-1}, u_{k-1}) = (1 + u_{1,k-1})^y - (1 + x_{1,k-1})^z = 0$$

where:

$$d_1^i = (c_1 + c_2(\bar{\alpha} + x_{2,i})), \; d_2^i = \bar{P}(1 + x_{1,i})(1 + \bar{\alpha} + x_{2,i})$$

Here y and z are the smallest integers such that $y/z = \gamma$. Note that the system dynamics are represented by rational vector fields. The multiplier

$w(x, u)$ is set equal to the denominator of $\Delta V(x)$, so that $w(x, u)\Delta V(x)$ is a polynomial. For the region of interest in the system state space, the denominator of $\Delta V(x)$ is always positive, so that $w(x, u)$ is always positive, as required.

A domain of attraction using sum of squares decomposition can be found, as outlined in Section 5. In order to solve the problem, the sum of squares Toolbox for Matlab (SOSTOOLS, (Prajna *et al.*, 2002)) is used. This software package automates the conversion from the sum of squares program to SDP, calls the SDP solver (SeDuMi, (Sturm, 1999)), and converts the SDP solution back to the form of the original sum of squares program.

Using a candidate Lyapunov function $V_{k-1}(x) = x_{1,k-1}^2 + 0.135x_{2,k-1}^2$ a linesearch of Ψ_{max} for the SOSP given in Section 5.3 yields a Ψ_{max} of 0.21. Figure 3 shows the phase plot for the system with the level curve corresponding to $V(x) = 0.21$. This corresponds to a region of attraction guaranteed through use of the technique outlined in Section 5. The shaded area shows the typical operating range of the HCCI engine. This method guarantees stability over the vast majority of that region. Physically this means that regulation about the desired operating point is guaranteed, even when the system is perturbed away from the local equilibrium region.

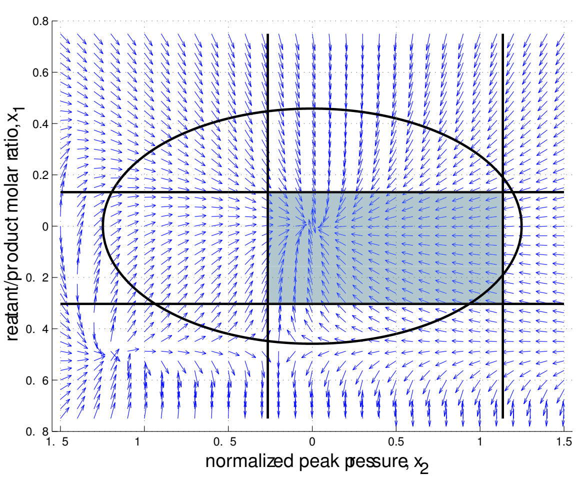

Fig. 3. Vector plot (only direction shown): shaded region is typical operation region

7. CONCLUSION

Nonlinear analysis tools have multiple applications in the realm of HCCI control. Contraction analysis can be used to show stable tracking of desired valve movements with a laboratory VVA system. Closed-loop stability of the HCCI combustion process can be shown by estimating the domain of attraction through the use of sum of squares decomposition, the Positvstellensatz and convex optimization. Together, these approaches guarantee stability of the combined valve/combustion control approach.

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